Gauss-Manin connection in noncommutative geometry

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Abstract. This paper is a review of results on the Gauss-Manin connection in noncommutative geometry. The Gauss-Manin connection in periodic cyclic homology was introduced by Ezra Getzler in 1991, then generalized to a superconnection by the author in a joint work with Dolgushev and Tamarkin. The key to these constructions is the Cartan calculus in noncommutative geometry. The original results of this article, namely the comparisons between the noncommutative Cartan calculus and its classical version, are contained in Section 5. The rest of the paper is mostly a review of results from **[13]**, **[9]**, **[27]**, **[33]**, although the approach is somewhat new.

To Ezra on his sixtieth birthday

1. Introduction

In **[25]**, Yu. I. Manin, then nineteen years old, considered the following question. Let $\{X_s | s \in S\}$ be a family of algebraic curves over $\mathbb{C}$. Fix an element γ in homology of X_s where s is a point of S. Let ω_s be an algebraic differential on X_s that depends on s algebraically. Locally in s, we can identify homologies of different fibers. Therefore, locally in s, the period $s \mapsto \int_\gamma \omega_s$ over a cycle γ is a function in s. It turns out that this function satisfies the Picard-Fuchs differential equation. Why is that so?

Manin's answer: the period actually depends on the class of ω_s modulo exact differentials. But the space of such classes is finite dimensional. So, when you start to differentiate the period, you will eventually get a linear relation, very much the same way as a matrix has to satisfy a polynomial equation.

But the space of classes of differentials depends on s. How do you differentiate a function with values in such a space? Manin provided an answer as follows. Let $\mathfrak{X}$ be the total space of the family, with X_s being fibers of the morphism $\mathfrak{X} \to S$. We work Zariski locally in $\mathfrak{X}$, actually passing to fields K and R of rational functions on S and $\mathfrak{X}$ respectively. A derivation of K can always be extended to a derivation of R. Manin gives a way to differentiate using such an extension, and then shows that the result does not depend on an extension.

Soon after, Grothendieck and then Katz and Oda re-interpreted and generalized Manin's construction in terms of the following. The assignment $s \mapsto H^\bullet_{\mathrm{DR}}(X_s)$ is actually a vector bundle on the base, and it admits a flat connection (the Gauss-Manin connection). This is explained by the Cartan calculus on differential forms.

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Namely, a vector field ξ defines the operator $L_\xi : \Omega_X^\bullet \to \Omega_X^\bullet$ as well as the operator $\iota_\xi : \Omega_X^\bullet \to \Omega_X^{\bullet-1}$ satisfying relations

$$\iota_\xi^2 = 0;\ d\iota_\xi + \iota_\xi d = L_\xi \tag{1.1}$$

Now replace X by an associative (possibly noncommutative) algebra A. De Rham cohomology gets replaced by periodic cyclic homology $\mathrm{HC}_\bullet^{\mathrm{per}}(A)$. In **[13]**, Ezra Getzler constructed a flat connection in the bundle $s \mapsto \mathrm{HC}_\bullet^{\mathrm{per}}(A_s)$ for a family of algebras over a smooth base S. This is based on the following generalization of the Cartan calculus. Let $C_\bullet(A)$ and $C^\bullet(A)$ be the standard chain and cochain complexes computing the Hochschild homology $\mathrm{HH}_\bullet(A)$ and the Hochschild cohomology $\mathrm{HH}^\bullet(A)$ **[3]**, **[23]**. For $\varphi \in C^m(A)$ there are two operators

$$L_\varphi : \mathrm{C}_\bullet(A) \to \mathrm{C}_{\bullet-m}(A);\ \iota_\varphi : \mathrm{C}_\bullet(A) \to \mathrm{C}_{\bullet-m+1}(A) \tag{1.2}$$

Those operators induce pairings at the level of Hochschild (co)homology that satisfy the standard Cartan relations **[10]**.

It is natural to ask what replaces the Cartan relations at the level of Hochschild complexes rather than at the level of (co)homology. Let us try to predict the answer. For any algebra A and any algebra endomorphism $f : A \to A$ we define a complex

$$\mathrm{TR}_A(f) = C_\bullet(A, {}_f A) \tag{1.3}$$

(Here ${}_f A$ is A viewed as an A-bimodule, with the action $a_0 \cdot a \cdot a_1 = f(a_0) a a_1$ and the right hand side of (1.3) is the Hochschild complex with coefficients in this bimodule. Roughly, this is a noncommutative analog of differential forms on the space of fixed points of an endomorphisms of a variety). By functoriality, there are two endomorphisms of $\mathrm{TR}_A(f)$: id_* and f_*. We predict that they should be chain homotopic. In fact one can deduce that from the fact that they coincide at the level of

$$\mathrm{tr}_A(f) = \mathrm{HH}_0(A, {}_f A) = {}_f A/[A, {}_f A] = A/\langle f(a_0)a_1 - a_1 a_0 \rangle \tag{1.4}$$

(here $\langle\rangle$ stands for linear span). So there should be an operator

$$B_f : \mathrm{TR}_A(f) \to \mathrm{TR}_A(f)[1] \tag{1.5}$$

satisfying

$$[B_f, b_f] = f_* - \mathrm{id}_* \tag{1.6}$$

(here b_f is the differential on the complex $\mathrm{TR}_A(f)$). Now, B_f^2 is a natural transformation $\mathrm{TR}_A(f) \to \mathrm{TR}_A(f)[2]$; we expect it to be chain homotopic to zero. Proceeding by induction, we expect to have natural operators $B_f^{(n)} : \mathrm{TR}_A(f) \to \mathrm{TR}_A(f)[1+2n], n \geq 0$, such that $B_f = B_f^{(0)}$ and

$$(b_f + u\mathbb{B}_f)^2 = f_* - \mathrm{id}_* \tag{1.7}$$

where u is a formal variable of cohomological degree 2 and

$$\mathbb{B}_f = \sum_{n=0}^{\infty} u^n B_f^{(n)}$$

This would have the following consequences. First, put $f = \mathrm{id}$. Put $\mathrm{TR}(A) = \mathrm{TR}_A(\mathrm{id})$, $b = b_{\mathrm{id}}$ and $\mathbb{B} = \mathbb{B}_{\mathrm{id}}$. Then (1.7) tells us that $\mathbb{B}$ is a differential on

$\mathrm{TR}(A)[[u]]$. And indeed, such a complex does exist; it is the negative cyclic complex $\mathrm{CC}^-_\bullet(A)$. Its localization $\mathrm{TR}(A)((u))$ is the periodic cyclic complex computing $\mathrm{HC}^{\mathrm{per}}_\bullet(A)$ as mentioned above.

Second, let D be a derivation of A (this is a noncommutative analog of a vector field on a variety). Applying the above formally to $f = \exp(D)$, we get natural operators

$$I_{D^n} : \mathrm{TR}(A)[[u]] \to \mathrm{TR}(A)[[u]][-1] \tag{1.8}$$

such that for

$$I(D) = \sum_{n=1}^{\infty} \frac{1}{n!} I_{D^n}$$

we have

$$(b + u\mathbb{B} + I(D))^2 = \exp(L_D) - 1 \tag{1.9}$$

Here $L_D = D_*$ is the action of D on $\mathrm{TR}(A)[[u]]$. This is equivalent to

$$[b + u\mathbb{B}, I_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} I_{D^k} I_{D^{n-k}} = L_D^n \tag{1.10}$$

Note that the above equation has no denominators and in fact holds over $\mathbb{Z}$. Note also that in reality, on the standard Hochschild complex we already have $B^2 = 0$ so we can put $\mathbb{B} = B$.

Let us compare this to the classical Cartan calculus. There, we already have $\iota_\xi^2 = 0$. So we can put

$$J(\xi) = \iota_\xi; \quad b = 0; \ \mathbb{B} = d \tag{1.11}$$

and rewrite the Cartan relations as

$$(b + u\mathbb{B} + J(\xi))^2 = uL_\xi \tag{1.12}$$

The discrepancy between the classical Cartan relations (1.12) and the noncommutative Cartan relations (1.9) can be removed (Lemma 4.5). Therefore, when A is commutative, we have two generalized Cartan calculi: the classical one on forms, and the new one on Hochschild chains. But (in characteristic zero) one can compare the two via the HKR map. We show that this comparison agrees with the Cartan calculus structure (Theorem 5.1).

Now let $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$-algebras on an algebraic variety S. Assume that ∇ is a connection on $\mathcal{A}$. *In the case when the connection* ∇ *preserves the algebra structure*, this allows to construct a Getzler-Gauss-Manin superconnection as follows:

$$\nabla^{\mathrm{GM}} = b + u\mathbb{B} + L_\nabla - \sum_{n=1}^{\infty} \frac{u^{-n}}{n!} J(R^n)$$

where $R = \nabla^2$. Note that the formula makes sense in characteristic zero when R is nilpotent (i.e. when the base is smooth).

But the above condition on ∇ is too restrictive, even locally in S. In the classical case, the interpretation of the Gauss-Manin connection due to Grothendieck and Katz-Oda starts with a connection defined locally in the total space of the family $\mathfrak{X}$ (cf. Section 2). But in the noncommutative case there is no such thing. Instead, following **[13]**, we look at the discrepancy together with the curvature:

$$R = \nabla m + \nabla^2$$

where m is the product on $\mathcal{A}$. This is a form with values in the Hochschilld cochain complex of $\mathcal{A}$. Hochschild cochains are noncommutative analogs of multivector fields, therefore we have to generalize to the noncommutative case the *extended Cartan calculus* between forms and multivectors. Here we do this using the approach to Cartan calculus originated by Khalkhali in **[21]** and **[22]**. Namely, the DG Lie algebra of Hochschild cochains is quasi-isomorphic to the algebra of coderivations of the cobar construction, and we can apply the dual version of the Cartan calculus as above to the DG coalgebra $\mathrm{Bar}(A)$ instead of A. Due to a theorem from **[31]** and **[11]**, the dual version of the periodic cyclic complex of $\mathrm{Bar}(A)$ is quasi-isomorphic to the periodic cyclic complex of A. We call it *the big periodic cyclic complex.* The above allows us to construct the Gauss-Manin superconnection on the big complex and prove its main properties (Theorem 8.5).

REMARK 1.1. The result holds not only in characteristic zero but also over the p-adics, provided the differential of the product is p-adically small. Moreover, the deviation of the multiplication from associativity only needs to be p-adically small, which allows to define a noncommutative crystalline complex of an algebra over $\mathbb{F}_p$. This is done in **[27]** and **[33]**; we plan to extend this paper's approach to the p-adic case elsewhere.

It would be interesting to investigate convergence of our formulas in other topologies. Note that those formulas involve exponentials which suggest some convergence. However, those are exponentials of the form $\exp(\frac{1}{u}S)$ where u is a formal parameter. This means that one can hope for convergence in some refined versions of the periodic cyclic complex in which some convergence property on chains $\sum_{j=-m}^{\infty} u^j c_j$ is imposed (cf. e.g. **[7]**, **[26]**).

REMARK 1.2. As shown in **[16]**, in characteristic zero the periodic cyclic complex $(C_\bullet(A)((u)), b+uB)$ can be replaced by the double complex $(\Omega_A^\bullet((u)), d+u\iota_\Delta)$. Here $\Omega^\bullet$ stands for noncommutative forms, d is the De Rham differential, and ι_Δ the Ginzburg-Schedler differential. For any derivation ξ of A one defines L_ξ and ι_ξ in a straightforward way, subject to strict Cartan relations (1.1). Passing from A to its cofibrant resolution, we get another version of a big periodic cyclic complex on which the Gauss-Manin superconnection can be constructed. We do not know how this can be adapted to the p-adic situation, although it might be interesting to investigate.

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2. The Gauss-Manin connection in algebraic geometry

We recall the classical facts about the Gauss-Manin connection **[25]**, **[17]**, **[20]**. Our exposition follows **[24]**.

2.1. Calc 1. Let $p : X \to S$ be a smooth morphism of smooth algebraic varieties over $\mathbb{C}$. The structure sheaf $\mathcal{O}_{X/S}$ is automatically a sheaf of $p^{-1}\mathcal{O}_S$-algebras; $\Omega^1_{X/S}$ denotes the module of Kähler differentials over $\mathcal{O}_S$; $\mathcal{T}_{X/S}$ is the Lie algebra of $\mathcal{O}_S$-derivations of $\mathcal{O}_{X/S}$. Let

$$\Omega^\bullet_{X/S} = \wedge^\bullet_{\mathcal{O}_{X/S}} \Omega^1_{X/S}$$

be the differential graded algebra of relative differential forms on X/S. The Lie algebra $\mathcal{T}_{X/S}$ acts on $\Omega^\bullet_{X/S}$ by Lie derivatives; the Abelian Lie algebra $\mathcal{T}_{X/S}[1]$ acts on the same sheaf by contractions. We denote these actions by

$$\xi \mapsto \mathrm{Lie}_\xi$$

and

$$\xi \mapsto \iota_\xi$$

for a vector field ξ. These operators satisfy the classical Cartan identities

$$[\mathrm{Lie}_\xi, \mathrm{Lie}_\eta] = \mathrm{Lie}_{[\xi,\eta]},\ [\mathrm{Lie}_\xi, \iota_\eta] = \iota_{[\xi,\eta]},\ [\iota_\xi, \iota_\eta] = 0\ \forall \xi, \eta \in \mathcal{T}_{X/S}. \tag{2.1}$$

The De Rham differential, $d_{X/S}$, satisfies

$$[d_{X/S}, d_{X/S}] = 0,\ [d_{X/S}, \iota_\xi] = \mathrm{Lie}_\xi,\ [d_{X/S}, \mathrm{Lie}_\xi] = 0. \tag{2.2}$$

To conclude:

LEMMA 2.1. *The formula $\xi + \eta\epsilon \mapsto \mathrm{Lie}_\xi + \iota_\eta$ defines an action of the differential graded Lie algebra $(\mathcal{T}_{X/S}[\epsilon], \frac{\partial}{\partial\epsilon})$ on the differential graded algebra $(\Omega^\bullet_{X/S}, d_{X/S})$. Here ϵ is a formal parameter of degree -1 and square zero.*

2.2. Given the morphism $p : X \to S$ as in Section 2.1, define a connection to be a linear map

$$\nabla : \mathcal{O}_{X/S} \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{O}_{X/S} \tag{2.3}$$

such that

$$\nabla(fg) = f\nabla(g) + g\nabla(f), \nabla(h) = d_S h\ \forall f, g \in \mathcal{O}_{X/S}, h \in \mathcal{O}_S, \tag{2.4}$$

where $d_S h$ is the differential of h. Geometrically, ∇ is a rule that lifts a vector field on S to that on X/S. Such connections exist at least locally on X.

The curvature of ∇ is an element

$$R = \frac{1}{2}[\nabla, \nabla] \tag{2.5}$$

of $\Omega^2_S \otimes_{\mathcal{O}_S} \mathcal{T}_{X/S}$.

We will now extend the scalars from $\mathcal{O}_S$ to the entire de Rham complex $\Omega^\bullet_S$, thus replacing $\Omega^\bullet_{X/S}$ with $\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}$. The connection ∇ extends to a derivation Lie_∇ of degree one of $\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}$ satisfying

$$\mathrm{Lie}_\nabla \alpha = d_S \alpha,\ \alpha \in \Omega^\bullet_S \tag{2.6}$$

For $\xi \in \Omega^\bullet_S \otimes_{\mathcal{O}_{X/S}} \mathcal{T}_{X/S}$, Lie_ξ and ι_ξ define derivations of $\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}$; we have

$$\mathrm{Lie}^2_\nabla = \mathrm{Lie}_R \tag{2.7}$$

Lemma 2.1 is still valid if we replace the Lie algebra $\mathcal{T}_{X/S}$ by the differential graded Lie algebra $\Omega^\bullet_S \otimes_{\mathcal{O}_S} \mathcal{T}_{X/S}$.

The Cartan identities (2.1) and (2.2) allow us to flatten Lie_∇. Namely, define the derivation

$$D_\nabla = \mathrm{Lie}_\nabla + d_{X/S} - \iota_R \in \mathrm{End}_{\mathbb{C}}(\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S})$$

Lemma 2.2. *The pair* $(\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}, D_\nabla)$ *is a complex, i.e.,*

$$D^2_\nabla = 0.$$

Proof. The result follows from the following computation:

$$\begin{aligned} & 2D^2_\nabla = [D_\nabla, D_\nabla] = \\ & [\mathrm{Lie}_\nabla, \mathrm{Lie}_\nabla] + [d_{X/S}, d_{X/S}] + [\iota_R, \iota_R] \\ + \quad & 2[\mathrm{Lie}_\nabla, d_{X/S}] - 2[\mathrm{Lie}_\nabla, \iota_R] - 2[d_{X/S}, \iota_R] = \\ & 2\mathrm{Lie}_R - 2\iota_{[\nabla,R]} - 2\mathrm{Lie}_R = 0, \end{aligned}$$

which consists in a repeated application of (2.1, 2.2), including

$$[\mathrm{Lie}_\nabla, \iota_R] = \iota_{[\nabla,R]} = 0$$

thanks to the Bianchi identity $[\nabla, R] = 0$. □

Various choices of ∇ give rise to canonically isomorphic complexes:

Lemma 2.3. *The map*

$$\exp(\iota_{\nabla_2 - \nabla_1}) : (\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}, D_{\nabla_1}) \to (\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}, D_{\nabla_2})$$

is an isomorphism of complexes.

Proof. It is clear that the operator $\iota_{\nabla_2 - \nabla_1} \in \mathrm{End}_{\mathbb{C}}(\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S})$ is nilpotent and is an even derivation of the algebra $\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}$. Therefore it defines an automorphism of $\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}$. What we need to check then is that

$$\exp(ad_{\iota_{\nabla_2 - \nabla_1}})(D_{\nabla_1}) = D_{\nabla_2},$$

where $ad_{\iota_{\nabla_2 - \nabla_1}}$ means the operation of taking the bracket, $[\iota_{\nabla_2 - \nabla_1}, .]$. Indeed, let A be an element of $\Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{T}_{X/S}$. Then

$$\exp(ad_{\iota_A})(\mathrm{Lie}_\nabla) = \mathrm{Lie}_\nabla - \iota_{\nabla A}; \tag{2.8}$$

$$\exp(ad_{\iota_A})(d_{X/S}) = d_{X/S} + \mathrm{Lie}_A + \frac{1}{2}[\iota_A, \mathrm{Lie}_A] \ = d_{X/S} + \mathrm{Lie}_A - \frac{1}{2}\iota_{[A,A]}; \tag{2.9}$$

Also $[\iota_A, \iota_R] = 0$, so

$$\exp(ad_{\iota_A})(d_{X/S} + \mathrm{Lie}_\nabla + \iota_R) = d_{X/S} + \mathrm{Lie}_\nabla - (\iota_R + \iota_{\nabla A} + \frac{1}{2}\iota_{[A,A]}) \tag{2.10}$$

But $R + \nabla A + \frac{1}{2}[A, A]$ is the curvature of the connection $\nabla + A$. □

2.3. Comparison with Katz-Oda. The complex $(\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}, D_\nabla)$ is filtered, $F^p(\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}) = (\Omega^{\geq p}_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S})$. It is easy to see that, locally in X, a connection ∇ identifies $(\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}, D_\nabla)$ with the absolute De Rham complex $(\Omega^\bullet_X, d)$. The filtration gets identified with the filtration from **[20]**. We see that our construction reproduces the Katz-Oda construction of the Gauss-Manin connection.

2.4. Globalization. We will now show that Lemmas 2.2 and 2.3 imply the existence of a globally defined complex; such a complex is not unique, but different such complexes are canonically isomorphic.

The variety S being smooth, it can be covered by open subsets S_α étale over $\mathbb{C}^n$, hence each having a coordinate system $\{u_i, \partial_{u_i}\}$, $\partial_{u_i}(u_j) = \delta_{ij}$. The morphism $p : X \to S$ being smooth, the preimage $p^{-1}(S_\alpha)$ has an open cover $\{U_{\alpha\beta}\}$ so that each $U_{\alpha\beta}$ has a coordinate system $\{\{u_i, x_j, \partial_{u_i}, \partial_{x_j}\}$. This means that over each $U_{\alpha\beta}$, the restriction $\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}|_{U_{\alpha\beta}}$ regarded as a sheaf over S_α carries a connection, $\nabla_{\alpha\beta}$.

To streamline the notation: we have a sheaf $\Omega \stackrel{\text{def}}{=} \Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}$ on X and an open cover $\{U_i\}$ of X; on each restriction $\Omega_i \stackrel{\text{def}}{=} \Omega|_{U_i}$ we have chosen a connection ∇_i and hence a differential $D_{\nabla_i} : \Omega_i \to \Omega_i$, Lemma 2.2. According to Lemma 2.3, over the intersection $U_i \cap U_j$ there is an isomorphism

$$G_{ji} \stackrel{\text{def}}{=} \exp\left(\iota_{\nabla_j - \nabla_i}\right) : (\Omega_i, D_{\nabla_i}) \to (\Omega_j, D_{\nabla_j}).$$

It is clear that on triple intersections we have the consistency condition:

$$G_{ki} = G_{kj} \circ G_{ji}.$$

Therefore, having fixed a cover $\{U_i\}$ and a bunch of connections $\{\nabla_i\}$, which we will denote simply by ∇, we obtain a new globally defined complex, Ω_∇, by tearing the original $\Omega^\bullet_S \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}$ apart and regluing the pieces $\{(\Omega_i, \nabla_i)\}$ using $\{G_{ij}\}$ as transition functions. This proves the existence.

Two distinct such complexes, Ω_∇ and $\Omega_{\nabla'}$ (attached to distinct collections ∇ and ∇') are canonically isomorphic. To see this, assume, as we can, that both are defined using the same open cover $\{U_i\}$. Then the collection of isomorphisms

$$\exp\left(\iota_{\nabla'_i} - \iota_{\nabla_i}\right) : \Omega_\nabla|_{U_i} \to \Omega_{\nabla'}|_{U_i}$$

defines an isomorphism of complexes $\Omega_\nabla \to \Omega_{\nabla'}$.

3. Noncommutative calculus

We closely follow **[27]** and **[28]**.

3.1. Cyclic complexes. For an associative unital algebra A over a commutative unital ring k, define

$$C_\bullet(A) = A \otimes (A/k)^{\otimes \bullet} \tag{3.1}$$

$$b : C_\bullet(A) \to C_{\bullet-1}(A);\ B : C_\bullet(A) \to C_{\bullet+1}(A); \tag{3.2}$$

$$b(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes a_n + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1}; \tag{3.3}$$

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_i (-1)^{i(n-i+1)} 1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots a_{i-1}. \tag{3.4}$$

One has

$$b^2 = 0;\ bB + Bb = 0;\ B^2 = 0. \tag{3.5}$$

The homology of the differential b is called the Hochschild homology of A and is denoted by $\mathrm{HH}_\bullet(A)$.

DEFINITION 3.1. The negative cyclic homology $HC^-_\bullet(A)$ (resp. periodic cyclic homology $HC^{per}_\bullet(A)$) of an associative algebra A is the homology of the complex

$$\mathrm{CC}^-_\bullet(A) = (C_\bullet(A)[[u]], b + uB); \tag{3.6}$$

resp.

$$\mathrm{CC}^{\mathrm{per}}_\bullet(A) = (C_\bullet(A)((u)), b + uB), \tag{3.7}$$

where u is a formal variable of degree -2.

3.1.1. *Motivation: the case of a commutative algebra.* Suppose that A is a commutative algebra and k contains the field of rational numbers. Define the Hochschild-Kostant-Rosenberg map

$$\mathrm{HKR} : C_\bullet(A) \to \Omega^\bullet_{A/k};\ a_0 \otimes \ldots \otimes a_n \mapsto \frac{1}{n!} a_0 da_1 \ldots da_n,\ n \geq 0 \tag{3.8}$$

It is easy to see that HKR intertwines b with 0 and B with d. Moreover it extends to

$$\mathrm{HKR} : CC^-_\bullet(A) \to (\Omega^\bullet_{A/k}[[u]], ud);\ \ CC^{\mathrm{per}}_\bullet(A) \to (\Omega^\bullet_{A/k}((u)), ud) \tag{3.9}$$

THEOREM 3.2. **[19]** *Suppose that A is a commutative regular algebra. Then* HKR *induces an isomorphism* $\mathrm{HH}_\bullet(A) \longrightarrow \Omega^\bullet_{A/k}$ *and quasi-isomorphisms*

$$\begin{aligned} CC^-_\bullet(A) &\longrightarrow (\Omega^\bullet_{A/k}[[u]], ud) \\ CC^{\mathrm{per}}_\bullet(A) &\longrightarrow (\Omega^\bullet_{A/k}((u)), ud) \\ (\wedge^\bullet T_{A/k}, 0) &\longrightarrow (C^\bullet(A), \delta). \end{aligned}$$

REMARK 3.3. In the case of $A = C^\infty(M)$, the same result holds after replacing algebraic tensor products with projective tensor products.

4. Operations on cyclic complexes

4.1. The Cartan calculus of derivations.

DEFINITION 4.1. For a derivation D of A and for $m \geq 1$, set

$$L_D(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} a_0 \otimes \ldots \otimes D(a_j) \otimes \ldots \otimes a_n$$

$$\iota_{D^m}(a_0 \otimes \ldots \otimes a_n) = a_0 D^m(a_1) \otimes a_2 \otimes \ldots \otimes a_n$$

$$S_{D^m}(a_0 \otimes \ldots \otimes a_n) = \sum_{j=1}^{n} (-1)^{nj} a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes L^m_D(a_1 \otimes \ldots \otimes a_{j-1})$$

$$I_D = \iota_D + uS_D.$$

We will often denote L_D simply by D.

LEMMA 4.2. *The following Cartan relations are satisfied:*

$$[b + uB, L_D] = 0;\ [L_D, L_E] = L_{[D,E]};\ [L_D, I_E] = I_{[D,E]};\ [b + uB, I_D] = uL_D \tag{4.1}$$

The missing Cartan relation

$$[I_D, I_E] = 0 \tag{4.2}$$

is true only at the level of homology. More precisely, PROPOSITION 4.3. *Put*

$$I_{D^n} = \iota_{D^n} + uS_{D^n},\ n \geq 1. \tag{4.3}$$

For all $n > 0$.

$$[b + uB, I_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} I_{D^k} I_{D^{n-k}} = uD^n \tag{4.4}$$

REMARK 4.4. For comparison, define in the commutative case the operations

$$J_{D^n} = \iota_D,\ n = 1;\ J_{D^n} = 0,\ n > 1 \tag{4.5}$$

then

$$[ud, J_D] = uD;\ [ud, J_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} J_{D^k} J_{D^{n-k}} = 0,\ n > 1; \tag{4.6}$$

and, as a consequence, setting $\mathcal{J}(D) = \sum_{n=1}^{\infty} \frac{1}{n!} J_{D^n}$, we get

$$(ud + \mathcal{J}(D))^2 = uD \tag{4.7}$$

On the other hand, the relations from Proposition 4.3 are equivalent to

$$(b + uB + \mathcal{I}(D))^2 = u(e^D - 1) \tag{4.8}$$

where again

$$\mathcal{I}(D) = \sum_{n=1}^{\infty} \frac{1}{n!} I_{D^n} \tag{4.9}$$

This seems to be another instance of the appearance of the inverse Todd series $\frac{e^D - 1}{D}$, as it often happens in the comparison between commutative and non-commutative contexts.

One can pass from $\mathcal{I}(D)$ to $\mathcal{J}(D)$ as follows.

LEMMA 4.5. *Define the Stirling numbers as the coefficients of the power series*

$$\sum_{k,l \geq 0} c_{k,l} x^k y^l = \sum_{n=1}^{\infty} \frac{1}{n!} y(y - x) \ldots (y - (n-1)x). \tag{4.10}$$

Let I_{D^n}, $n \geq 1$, *satisfy the relations from the proposition 4.3 and set*

$$\mathcal{J}(D) = \sum_{k,l \geq 0} c_{k,l} L_D^k I_{D^l}.$$

Then

$$(b + uB + \mathcal{J}(D))^2 = uL_D$$

EXAMPLE 4.6. Let $\mathcal{A}$ be a differential graded algebra whose differential we denote by $d_{\mathcal{A}}$. Let α be a derivation of degree one of $\mathcal{A}$ and set

$$R = (d_{\mathcal{A}} + \alpha)^2 = [d_{\mathcal{A}}, \alpha] + \frac{1}{2}[\alpha, \alpha].$$

Consider first the commutative case and the de Rham complex $\Omega^{\bullet}_{\mathcal{A}/k}$ with the differential $d_{\mathcal{A}} + d$. Then

$$D^u_\alpha = d_{\mathcal{A}} + ud - \frac{1}{u} \iota_R \tag{4.11}$$

is a differential on $\Omega^{\bullet}_{\mathcal{A}/k}((u))$.

Now, more generally, assume that, in addition to the Lie algebra $\mathrm{Der}(\mathcal{A})$, a collection of J_{R^n} acts on a complex with differential $b+uB$, subject to (4.6). Then, formally, set

$$\mathbf{D}^u_\alpha = d_{\mathcal{A}} + ud - \Psi(R) \text{ where } \Psi(R) = \sum_{n=1}^{\infty} \frac{u^{-n}}{n!} J_{R^n}. \tag{4.12}$$

One checks that $(\mathbf{D}^u_\alpha)^2 = 0$.

Let us now assume that instead of a collection of operators J_{R^n}, a collection of operators I_{R^n} acts subject equations (4.3). For example, the complex could be $\mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{A})$. Looking for $\Phi(R)$ such that

$$(d_{\mathcal{A}} + b + uB - \Phi(R))^2 = 0, \tag{4.13}$$

we find

$$\Phi(R) = \sum_{k,l} c_{k,l} R^l I_{R^k} \tag{4.14}$$

where

$$\sum_{k,l} c_{k,l} y^l x^k = (1 + \frac{y}{u})^{\frac{x}{y}} - 1 = \sum_{n=1}^{\infty} \frac{1}{n! u^n} x(x-y)\dots(x-(n-1)y) \tag{4.15}$$

We see that $\exp(\frac{x}{u})$ gets replaced by $(1+\frac{y}{u})^{\frac{x}{y}}$.

Both operators require a convergence condition. For example, one might assume that k contains the rationals, the image of R is inside an ideal of $\mathcal{A}$. Or, one assumes that the image of R is contained in $p\mathcal{A}$ where $p > 2$ is a prime. In both cases, $b + uB + d_{\mathcal{A}} - \Phi(R)$ is a well-defined differential on the periodic cyclic complex completed with respect to the filtration induced by powers of the ideal (in the second case, this means p-adic completion); cf. **[27]**, **[33]**.

Remark 4.7. As shown in **[1]**, given a manifold with an action of a Lie group and a closed odd equivariant differential form, the space of equivariant forms acquires a generalized Cartan calculus structure as in (4.6). There are more examples of such generalized calculus: one in **[2]** (again as in (4.6)), another in **[15]** (as in (4.4)). The author is grateful to Anton Alekseev for pointing this out.

5. Compatibility with the Hochschild-Kostant-Rosenberg map

As we have seen above, when A is a commutative algebra and D is its derivation, one can define formal operators

$$\mathcal{J}(D) = \sum_{n=1}^{\infty} \frac{1}{n!} J_{D^n}$$

in two ways. One, on $\Omega^\bullet_{A/k}[[u]]$, is defined in Remark 4.4. The other, on $C_\bullet(A)[[u]]$, is constructed by Lemma 4.5 from (4.3). Both satisfy

$$(b + uB + \mathcal{J}(D))^2 = uD$$

where $b + uB$ on forms stands for ud.

THEOREM 5.1. *Suppose that A is commutative. Let k be of characteristic zero. Let D be a derivation of A. There exist natural (homogeneous of degree m in D) morphisms of degree zero*

$$\mathbf{HKR}_{D^m} : C_\bullet(A) \to \Omega^\bullet_{A/k}$$

such that $\mathbf{HKR}_0 = \mathrm{HKR}$ *is the quasi-isomorphism given in Theorem 3.2 and*

$$(ud + \mathcal{J}(D))\mathbf{HKR}(D) = \mathbf{HKR}(D)(b + uB + \mathcal{J}(D))$$

where

$$\mathbf{HKR}(D) = \sum_{n=0}^{\infty} \mathbf{HKR}_{D^n}$$

The proof occupies the rest of Section 5.

5.1. Natural operators $\mathcal{J}(D)$ on forms. Let us look at all possible natural systems of operators J_{D^n} on forms that satisfy (4.6). Any such system is of the form

$$\mathcal{J}(D) = A(D)\iota_D + B(D)ud \tag{5.1}$$

The condition (4.6) becomes

$$A(B+1) = 1$$

We claim that any two such $\mathcal{J}_1(D)$ and $\mathcal{J}_2(D)$ with $A(0) = B(0)$ are equivalent, meaning that there exists an invertible formal operator $F(D)$ such that

$$(ud + \mathcal{J}_2(D))F(D) = F(D)(ud + \mathcal{J}_1(D)) \tag{5.2}$$

The only natural candidate for such an operator is

$$F(D) = P(D) + Q(D)d\iota_D \tag{5.3}$$

where P and Q are some power series to be determined. The condition (5.2) becomes

$$-(P + DQ)A_1 + A_2P = 0;\;\; B_2(P + DQ) = PB_1 - QD \tag{5.4}$$

Since $B_i = A_i^{-1} - 1$, those two equations are the same. We can start with any invertible $P(D)$ and find $Q(D)$ uniquely out of the first equation. We will be interested in a special choice of $\mathcal{J}(D)$.

LEMMA 5.2. *Define*

$$I_D = \iota_D + \frac{D}{2}ud;\; I_{D^n} = \frac{D^n}{n+1}ud,\, n \geq 2. \tag{5.5}$$

and

$$\mathcal{J}_Q(D) = \frac{\log(1+D)}{D}\iota_D + (\frac{D}{\log(1+D)} - 1)ud \tag{5.6}$$

In other words:

$$\frac{1}{n!}J_{Q,D^n} = \frac{(-1)^{n-1}}{n}D^{n-1}\iota_D + c_nD^nud \tag{5.7}$$

where the c_n are defined by the generating function

$$1 + \sum_{n=1}^{\infty} c_nz^n = \frac{z}{\log(1+z)} \tag{5.8}$$

Then the I_{D^n} satisfy (4.4) *(with ud instead of $b+uB$); the J_{D^n} are the result of applying the construction of Lemma 4.5. In particular, the J_{D^n} satisfy* (4.6).

(In other words: (5.5) provides the simplest family of I_{D^n} on forms; applying the construction of Lemma 4.5 to it, we get a family of J_{D^n}, but this is not the simplest one possible).

PROOF.

$$[ud, I_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} I_{D^k} I_{D^{n-k}} = n([\iota_D, \frac{u}{n} dD^{n-1}] = uD^n;$$

$$\sum_{k+l=n} c_{k,l} L_D^k I_{D^l} = c_{n-1,1}\iota_D D^{n-1} + \sum_{k+l=n} c_{k,l} D^k \frac{D^l}{l+1} =$$
$$\frac{(-1)^{n-1}}{n} D^{n-1}\iota_D + \frac{u}{n!}\frac{1}{D}\int_0^D y(y-D)\dots(y-(n-1)D)dy$$

Note also

$$\sum_{n=1}^{\infty} \frac{1}{xn!}\int_0^x y(y-x)\dots(y-(n-1)x)dy = \frac{1}{x}\int_0^x (1+x)^{\frac{y}{x}} dy - 1 = \frac{x}{\log(1+x)} - 1$$

This proves our statement. $\square$

5.2. The operator HKR(D)**.** We will construct a formal operator

$$\mathrm{HKR}(D) = \mathrm{HKR} + \sum_{n=1}^{\infty} \mathrm{HKR}_{D^n} \tag{5.9}$$

such that HKR_{D^n} is a homogeneous map of degree n from $\mathrm{Der}(A)$ to operators $C_\bullet(A,A) \to \Omega^\bullet_{A/k}$ satisfying

$$[b + uB + \mathcal{I}(D), \mathrm{HKR}(D)] = 0 \tag{5.10}$$

Here $\mathcal{I}(D)$ on Hochschild chains is defined in Proposition 4.3 and on forms in (5.5). Having done that, we will be able to construct $\mathrm{HKR}_Q(D)$ as in Theorem 5.1 but with $\mathcal{J}_Q(D)$ instead of $\mathcal{J}(D)$:

$$\mathrm{HKR}_{Q,D^n} = \sum_{k+l=n} c_{k,l} D^k \mathrm{HKR}_{D^l}$$

Since $\mathcal{J}(D)$ and $\mathcal{J}_Q(D)$ are equivalent, this will prove the theorem.

The obstructions to existence of HKR(D) lie in the cohomology of an easy-to-compute complex. We will show that those obstructions all vanish.

5.2.1. *The complex of operations.* Let $n \geq 0$. Consider a collection $m_i \geq 0$ and $\epsilon_i = 0$ or 1, $i = 0, \dots, n$, such that for any $i > 0$, m_i and ϵ_i are not both equal to zero. Define the morphism $C_\bullet(A) \to \Omega^\bullet_{A/k}$

$$a_0 \otimes \dots \otimes a_n \mapsto d^{\epsilon_0} D^{m_0}(a_0) \dots d^{\epsilon_n} D^{m_n}(a_n) \tag{5.11}$$

Denote this operation by $\eta^{\epsilon_0} t^{m_0} \otimes \dots \otimes \eta^{\epsilon_n} t^{m_n}$.

LEMMA 5.3. *The linear span of all operations* (5.11) *with the differential $\circ b$ is isomorphic to the Hochschild cochain complex of the coalgebra $k[\eta, t]$ where $|\eta| = -1$, $|t| = 0$, $\eta^2 = 0$ and the comultiplication is the algebra morphism such that $t \mapsto t \otimes 1 + 1 \otimes t$, $\eta \mapsto \eta \otimes 1 + 1 \otimes \eta$.*

PROOF. Straightforward. $\square$

We denote our coalgebra by C and the complex of operations by $C^\bullet(C)$. Inside this complex there is a subcomplex $\mathrm{Cobar}(C)$ spanned by operations for which $m_0 = \epsilon_0 = 0$.

The cohomology of $C^\bullet(C)$ is straightforward. First, $C^\bullet(C) \xrightarrow{\sim} C^\bullet(k[t]) \otimes C^\bullet(k[\eta])$ by means of the coshuffle product: $t^{m_0}\eta^{\epsilon_0} \otimes \ldots \otimes t^{m_n}\eta^{\epsilon_n}$ maps to zero if for some $i > 0$ both m_i and ϵ_i are nonzero; otherwise, it maps to

$$(t^{m_0} \otimes t^{m_{i_1}} \otimes \ldots \otimes t^{m_{i_k}}) \otimes (\eta^{\epsilon_0} \otimes \eta^{\epsilon_{j_1}} \otimes \ldots \otimes \eta^{\epsilon_{j_{n-k}}})$$

where $m_{j_p} > 0$ for all p, $\epsilon_{j_q} = 1$ for all q, $i_1 < \ldots < i_k$, $j_1 < \ldots < j_{n-k}$, and $\{1, \ldots, n\} = \{i_1, \ldots, i_k, j_1 \ldots, j_{n-k}\}$. Furthermore: $C^\bullet(k[t])$ projects quasi-isomorphically to its cohomology which is spanned over k by t^m and $t^m \otimes t$, $m \geq 0$. The projection sends a monomial t^{m_0} or $t^{m_0} \otimes t$ to itself, and any other monomial $t^{m_0} \otimes \ldots \otimes t^{m_n}$ to zero. Note also that the differential on $C^\bullet(k[\eta])$ is zero.

We have constructed an explicit quasi-isomorphism

$$C^\bullet(C) \to (\oplus_{m\geq 0}(kt^m \oplus kt^m \otimes t)) \otimes (\oplus_{n\geq 0}(k1 \otimes \eta^{\otimes n} \oplus k\eta \otimes \eta^{\otimes n})) \tag{5.12}$$

which restricts to

$$\mathrm{Cobar}(C) \to (\oplus_{m\geq 0}(k1 \oplus k1 \otimes t)) \otimes (\oplus_{n\geq 0}(k1 \otimes \eta^{\otimes n}) \tag{5.13}$$

5.2.2. *The existence for* $\mathrm{HKR}(D)$. First observe that $[\mathrm{HKR}, \iota_D]$ maps to zero by (5.12). Therefore there is h_D in $\mathrm{Cobar}(C)$ such that $[\mathrm{HKR}, \iota_D] + h_D \circ b = 0$. We want to construct HKR_{D^m} such that

$$[\mathrm{HKR}, \iota_{D^m}] + \sum_{k=1}^{m-1} \binom{m}{k} [\mathrm{HKR}_{D^k}, \iota_{D^{m-k}}] = -\mathrm{HKR}_{D^m} \circ b \tag{5.14}$$

for all m. Assume we have constructed all D^k for $k < n$. Then the left hand side of (5.14) is an operation in $\mathrm{Cobar}(C)$ that vanishes under (5.13) (indeed, it has weight > 1 in t). This shows that HKR_{D^n} exists.

We claim that

$$\mathrm{HKR}(D) = \mathrm{HKR} + \sum_{n\geq 1} \frac{1}{n!}\mathrm{HKR}_{D^n} \tag{5.15}$$

is the one we are looking for (i.e. that no higher terms in u are needed; note also that, if needed, they would not be available because there are no natural operations $C_\bullet \to \Omega^\bullet$ of degree bigger that one). To proceed, we do need some explicit knowledge of HKR_{D^m}.

Let h_{D^m} be the principal term of HKR_{D^m} in the following sense: it is the part involving only the monomials (5.11) where only one m_j is nonzero.We claim that

$$h_{D^m}(a_0 \otimes \ldots \otimes a_n) = \sum_{k=1}^{n} \frac{(n-k+m)!}{(n+m)!(n-k)!} a_0 da_1 \ldots dD^m(a_k) \ldots da_n \tag{5.16}$$

or equivalently
(5.17)

$$h_{D^m}(a_0 \otimes \ldots \otimes a_n) = \mathrm{HKR}(\sum_{k=1}^{n} \frac{n!(n-k+m)!}{(n+m)!(n-k)!} a_0 \otimes a_1 \ldots \otimes D^m(a_k) \otimes \ldots \otimes a_n)$$

To see that, write

$$h_{D^m}(a_0 \otimes \ldots \otimes a_n) = \sum_{k=1}^{n} c_j^m(n) a_0 da_1 \ldots dD^m(a_k) \ldots da_n \tag{5.18}$$

Now apply (5.14) to $a_0 \otimes \ldots \otimes a_{n+1}$. Compute the coefficients of $a_0 D^m(a_1) da_2 \ldots da_{n+1}$. For $m = 1$, we get

$$c_1^1(n) + \frac{1}{(n+1)!} - \frac{1}{n!} = 0 \tag{5.19}$$

For $m > 1$, we get

$$c_1^m(n) + mc_1^{m-1}(n+1) - \frac{1}{n!} = 0 \tag{5.20}$$

Computing the coefficients of $a_0 da_1 \ldots D^m(a_j) da_{j+1} \ldots da_{n+1}$ for $j > 1$, we get

$$c_j^1(n) - c_{j-1}^1(n) + \frac{1}{(n+1)!} = 0 \tag{5.21}$$

for $m = 1$ and

$$c_j^m(n) - c_{j-1}^m(n) + mc_1^{m-1}(n+1) = 0 \tag{5.22}$$

for $m > 1$. In other words, we have recursive relations (5.20) and (5.21) where $c_j^0(n) = \frac{1}{n!}$ (this may look at first a bit strange as compared to (5.18)). These relations have the unique solution

$$c_j^m(n) = \frac{(n+m-k)!}{(n+m)!(n-k)!} \tag{5.23}$$

Now we have to show that

$$[\mathrm{HKR}, S_{D^m}] + \sum_{k=1}^{m-1} \binom{m}{k} [\mathrm{HKR}_{D^k}, S_{D^{m-k}}] + [\mathrm{HKR}_{D^m}, B] = 0 \tag{5.24}$$

Let us show that the composition of the left hand side with b is zero. Denote

$$\iota(D) = \sum_{m=1}^{\infty} \iota_{D^m};\ S(D) = \sum_{m=1}^{\infty} S_{D^m}$$

We have

$$[b + \iota(D), \mathrm{HKR}(D)] = 0 \tag{5.25}$$

and

$$[b + \iota(D), B + S(D)] = e^D - 1$$

Therefore

$$[b + \iota(D), [B + S(D), \mathrm{HKR}(D)]] = 0 \tag{5.26}$$

Let us write

$$[B + S(D), \mathrm{HKR}(D)] = \sum_{n=1}^{\infty} A_{D^n} \tag{5.27}$$

By induction in m, we see that $[b, A_{D^m}] = 0$ for all m.

Next, let us show that A_{D^m} map to zero under (5.12). The only terms that do not automatically map to zero are $dD^m(a_0) da_1 \ldots da_n$, corresponding to the operation $\eta t^m \otimes \eta^{\otimes m}$. There are two places in (5.14) in which this may appear.

One is $-S_{D^m}\mathrm{HKR}$ in the first summand; the other is $h_{D^m}B$ in the last summand. Recall that

$$S_{D^m} = \frac{dD^m}{m+1}$$

on forms. Therefore the total coefficient of $\eta t^m \otimes \eta^{\otimes m}$ in the left hand side of (5.14) is

$$-\frac{1}{(m+1)n!} + \sum_{j=1}^{n+1} c_j^m(n+1)$$

But

$$\sum_{j=1}^{n} c_j^m(n) = \sum_{j=1}^{n} \frac{(n+m-j)!}{(n+m)!(n-j)!} = \frac{m!}{(n+m)!}\sum_{j=1}^{n}\binom{n+m-j}{m} =$$

$$\frac{m!}{(n+m)!}\sum_{k=0}^{n-1}\binom{m+k}{m} = \frac{m!}{(n+m)!}\binom{n+m}{m+1} = \frac{1}{(m+1)(n-1)!}$$

Therefore every A_{D^m} vanishes under (5.12). This means that it is in the image of the differential. But it corresponds to a linear combination of terms $\eta t^{m_0}\otimes\ldots\otimes\eta t^{m_n}$ in $C^\bullet(C)$ which cannot be in the image of the differential. Indeed, the differential preserves the weight in η and increases n by one. Thus, all A_{D^m} are equal to zero. This concludes the proof of Theorem 5.12.

6. Extended Cartan calculus, I

6.1. Algebras $\mathcal{A}_0$ and $\mathcal{A}_1$.

Definition 6.1. For any differential graded Lie algebra $\mathfrak{g}$, let $U^+(\mathfrak{g})$ be the kernel of the augmentation $U(\mathfrak{g}) \to k$. Let $\mathrm{Cobar}(U^+(\mathfrak{g}))$ be the free associative algebra generated by $U^+(\mathfrak{g})[-1]$ (the degree shift by one). We denote the free generator corresponding to $x \in U^+(\mathfrak{g})$ by (x). Define

$$\partial_{\mathrm{Cobar}}(x) = \sum (-1)^{|x^{(1)}|}(x^{(1)})(x^{(2)}) \tag{6.1}$$

where the comultiplication is defined by

$$\Delta x = \sum x^{(1)} \otimes x^{(2)}$$

In addition, the differential $d_{\mathfrak{g}}$ induces a differential on $\mathrm{Cobar}(U^+(\mathfrak{g}))$. Now define the dg algebra

$$\mathcal{A}_1(\mathfrak{g}) = U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(U^+(\mathfrak{g})) \tag{6.2}$$

as follows. It is an algebra over $k[u]$ generated by the DG subalgebra $(U(\mathfrak{g}), d_{\mathfrak{g}})$ and the subalgebra $\mathrm{Cobar}(U^+(\mathfrak{g}))$. The only additional relations are

$$[X, (x)] = (-1)^{|X|}(\mathrm{ad}_X(x)),\ X \in \mathfrak{g}, x \in U^+(\mathfrak{g}).$$

The differential acts as follows:

$$x \mapsto d_{\mathfrak{g}}x, x \in U(\mathfrak{g});\ (x) \mapsto (-d_{\mathfrak{g}}x) + \partial_{\mathrm{Cobar}}(x) + ux,\ x \in U^+(\mathfrak{g}). \tag{6.3}$$

Define also

$$\mathcal{A}_0(\mathfrak{g}) = U(\mathfrak{g}) \ltimes_0 \mathrm{Cobar}(\mathrm{Sym}^+(\mathfrak{g})) \tag{6.4}$$

in the same way as above, the differential being

$$x \mapsto d_{\mathfrak{g}}x, x \in U(\mathfrak{g});\ (x) \mapsto (-d_{\mathfrak{g}}x) + \partial_{\mathrm{Cobar}}(x) + uB_0(x) \tag{6.5}$$

where

$$B_0(x) = x,\ x \in \mathrm{Sym}^1(\mathfrak{g}); B_0(x) = 0, x \in \mathrm{Sym}^{>1}(\mathfrak{g})$$

Lemma 4.2 can be strengthened as follows.

PROPOSITION 6.2. *There is a natural action of $\mathcal{A}_1(\mathrm{Der}(A))$ on $\mathrm{CC}^-_\bullet(A)$ where $\mathfrak{g} = \mathrm{Der}(A)$ such that for any derivation D, D acts by L_D and (D^n) acts by I_{D^n}* **[32]**, **[9]**.

Let us define this action expicitly. Recall for $D \in \mathrm{Der}(A)$

$$L_D(a_1 \otimes \ldots \otimes a_n) = \sum_{j=1}^{n} a_1 \otimes \ldots \otimes D(a_j) \otimes \ldots \otimes a_n = \lambda_D(a_1 \otimes \ldots \otimes a_n) \tag{6.6}$$

(to reconcile this with our notation later on, we use two different letters for the same operator). Let $X_1, \ldots, X_m \in \mathrm{Der}(A)$. For $X = X_1 \ldots X_m \in U^+(\mathrm{Der}(A))$, we put

$$\lambda_X = \lambda_{X_1} \ldots \lambda_{X_m} \tag{6.7}$$

DEFINITION 6.3. For $X \in U^+(\mathrm{Der}(A)$

$$I_X = \iota_{\overline{X}} + uS_X \tag{6.8}$$

where

$$\overline{X_1 \ldots X_m} = X_1 \circ \ldots \circ X_m; \tag{6.9}$$

$$\iota_{\overline{X}}(a_0 \otimes \ldots \otimes a_n) = a_0 \overline{X}(a_1) \otimes a_2 \otimes \ldots \otimes a_n \tag{6.10}$$

$$S_X(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} (-1)^{nj} a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes \lambda_X(a_1 \otimes \ldots \otimes a_{j-1}) \tag{6.11}$$

LEMMA 6.4. *The above formulas define an action of $\mathcal{A}_1(\mathrm{Der}(A))$ on $\mathrm{CC}^-_\bullet(A)$.*

PROOF. This can be shown by direct verification **[32]**; it follows also from considerations as in the introduction (namely, the argument establishing (1.10)). □

7. Extended Cartan calculus, II

7.1. Noncommutative calculus of multivector fields and forms. Return to the case of a commutative algebra A. Recall that $\wedge^\bullet T_{A/k}[1]$ carries a graded Lie algebra structure. The action by operators ι_D, $D \in Der(A)$ extends to an action by contraction of multivectors (multivector fields) so that $\Omega^{-\bullet}$ is a graded module over the graded algebra $\wedge^\bullet T_{A/k}$. Set, for $\alpha \in \Lambda^m T_{A/k}$,

$$L_\alpha = [d, \iota_\alpha] : \Omega^\bullet_{A/k} \to \Omega^{\bullet - m+1}_{A/k} \tag{7.1}$$

THEOREM 7.1. *The operation of Lie-derivative L makes $\Omega^{\bullet-m+1}_{A/k}$ into a module over the Lie algebra $\wedge^\bullet T_{A/k}[1]$. The following identities hold.*

1. $[L_\alpha, L_\beta] = L_{[\alpha,\beta]}$;
2. $[L_\alpha, \iota_\beta] = (-1)^{|\alpha|-1} \iota_{[\alpha,\beta]}$;
3. $[\iota_\alpha, \iota_\beta] = 0$;
4. $[d, \iota_\alpha] = L_\alpha$.

Moreover,

$$\iota_{\alpha\beta} = \iota_\alpha \iota_\beta;\ L_{\alpha\beta} = L_\alpha \iota_\beta + (-1)^{|\alpha|} \iota_\alpha L_\beta \tag{7.2}$$

7.2. Hochschild cochain complex. The role of noncommutative multivector fields is played by Hochschild cochains.

DEFINITION 7.2. Let A be a unital associative algebra over k. Set

$$C^\bullet(A) = \mathrm{Hom}_k(A^{\otimes \bullet}, A) \tag{7.3}$$

with the differential $\delta : C^\bullet(A) \to C^{\bullet+1}(A)$ given by

$$\delta\phi(a_1, \ldots, a_{n+1}) = a_1 \phi(a_2, \ldots, a_{n+1})$$
$$+ \sum_{k=1}^{n} (-1)^k \phi(a_1, \ldots, a_k a_{k+1} \ldots a_{n+1}) + \phi(a_1, \ldots, a_n) a_{n+1}$$

The complex $(C^\bullet(A), \delta)$ computes the groups $Ext^\bullet_{A\otimes A^{op}}(A, A)$, usually called the Hochschild cohomology of A and denoted by $\mathrm{HH}^\bullet(A)$ or $\mathrm{HH}^\bullet(A, A)$.

Note that the space of cocycles in $C^1(A)$ is $\mathrm{Der}(A)$. There is a differential graded Lie algebra structure on $C^\bullet(A)[1]$ that extends the commutator of derivations (the Gerstenhaber bracket). For later use, let us be more explicit. Let $\phi \in C^k(A)$ and $\psi \in C^l(A)$. Set

$$\phi \circ \psi = \phi \circ (\psi \otimes id \ldots \otimes id) + (-1)^{|\psi|} \phi \circ (id \otimes \psi \otimes id \ldots \otimes id) +$$
$$\ldots + (-1)^{|\psi||\phi|} \phi \circ (id \otimes \ldots \otimes \psi)$$

and

$$[\phi, \psi] = \phi \circ \psi - (-1)^{|\psi||\phi|} \psi \circ \phi.$$

Here $|\cdot|$ refers to the Lie algebra degree, i. e., for $\phi \in C^k$, $|\phi| = k - 1$. $[\cdot, \cdot]$. The bracket is called the Gerstenhaber bracket and the following holds.

THEOREM 7.3. *$(C^\bullet(A)[1], [\cdot, \cdot], \delta)$ is a differential graded Lie algebra.*

7.3. Hochschild cochains and the bar construction. Recall the approach to Cartan calculus originated in **[21]** and **[22]**. For a graded k-module A, let $\mathrm{Bar}(A) = T(A[1])$ be the cofree coassociative coalgebra. As usual, we denote $a_1[1] \otimes \ldots \otimes a_n[1]$ by $(a_1| \ldots |a_n)$. An A_∞ structure on A is by definition a coderivation of degree $+1$ and of square zero of $\mathrm{Bar}(A)$. Any coderivation is uniquely determined by its composition with the projection onto $A[1]$. In other words, an A_∞ structure on A is a collection of k-linear maps $m_n : A^{\otimes n} \to A[2 - n]$ for $n \geq 1$. Any DG algebra is an A_∞ algebra if we put $m_1(a_1) = da_1$, $m_2(a_1, a_2) = (-1)^{|a_1|} a_1 a_2$ for $n = 2$, and $m_n = 0$ for $n > 2$.

LEMMA 7.4. *The DG Lie algebra* $\mathrm{Coder}(\mathrm{Bar}(A))$ *is isomorphic to the DG Lie algebra* $(C^\bullet(A)[1], [\cdot, \cdot], \delta)$ *from 7.3.*

7.4. Hochschild and cyclic complexes of coalgebras. For a counital DG coalgebra B, let

$$C^{\bullet}_{\mathrm{II}}(B) = \bigoplus_{n=0}^{\infty} B \otimes \bar{B}[-1]^{\otimes n} \tag{7.4}$$

Here $\bar{B}$ is the kernel of the counit $B \to k$. The above complex carries two differentials: one coming from the differential on B, the other defined in the dual way to the Hochschild differential B. Cf. [**8**], [**18**], [**33**]. We view $C^{\bullet}_{\mathrm{II}}(B)$ as a complex whose differential is the sum of these two differentials.

Remark 7.5. If we followed definition of the Hochschild complex $C_{\bullet}(A)$ of a DG algebra A and applied the dual construction to a coalgebra B, we would get the direct product totalization, not the direct sum. This is why we use the subscript II and use the term complex of the second kind (compare to [**30**]). Our definition is *not* invariant under quasi-isomorphisms of DG coalgebras (which is good because we are going to apply it to an acyclic coalgebra $\mathrm{Bar}(A)$).

Similarly, we define the differential B by a formula dual to the one on the chain complex. Then we define cyclic complexes of a DG coalgebra by

$$\mathrm{CC}^{\bullet}_{\mathrm{II}}(B) = (C^{\bullet}_{\mathrm{II}}(B)[[v^{-1}]], d_B + b + v^{-1}B) \tag{7.5}$$

$$\mathrm{CC}^{\bullet}_{\mathrm{per,II}}(B) = (C^{\bullet}_{\mathrm{II}}(B)((v^{-1})), d_B + b + v^{-1}B) \tag{7.6}$$

$$\mathrm{CC}^{\bullet}_{-,\mathrm{II}}(B) = (C^{\bullet}_{\mathrm{II}}(B)((v^{-1}))/v^{-1}C^{\bullet}_{\mathrm{II}}(B)[[v^{-1}]], d_B + b + v^{-1}B) \tag{7.7}$$

Finally, for any DG coalgebra B (not necessarily counital), we denote by B^+ the coalgebra with counit adjoined, and but

$$\mathrm{C}^{\bullet}_{?,\mathrm{II}}(B) = \ker(\mathrm{C}^{\bullet}_{?,\mathrm{II}}(B^+) \to \mathrm{C}^{\bullet}_{?,\mathrm{II}}(k)) \tag{7.8}$$

$$\mathrm{CC}^{\bullet}_{?,\mathrm{II}}(B) = \ker(\mathrm{CC}^{\bullet}_{?,\mathrm{II}}(B^+) \to \mathrm{CC}^{\bullet}_{?,\mathrm{II}}(k)) \tag{7.9}$$

where ? stands for $-$, per, or nothing.

Theorem 7.6. *(cf.* [**31**], [**11**]*). There is a natural chain of quasi-isomorphisms*

$$\mathrm{C}_{-\bullet}(A) \xrightarrow{\sim} \mathrm{C}^{\bullet}_{\mathrm{II}}(\mathrm{Bar}(A))$$
$$\mathrm{CC}^{-}_{-\bullet}(A) \xrightarrow{\sim} \mathrm{CC}^{\bullet}_{\mathrm{II}}(\mathrm{Bar}(A))$$
$$\mathrm{CC}_{-\bullet}(A) \xrightarrow{\sim} \mathrm{CC}^{\bullet}_{-,\mathrm{II}}(\mathrm{Bar}(A))$$

The proof will be given in Subsections 7.5 and 7.6 below.

7.5. Short chain complexes. For an algebra A, let $C^{\mathrm{sh}}(A)$ be the truncated Hochschild complex

$$0 \to C_1(A)/bC_2(A) \xrightarrow{b} C_0(A) \to 0 \tag{7.10}$$

Equivalently,

$$C^{\mathrm{sh}}_1(A) \xrightarrow{\sim} \Omega^1_A/[A, \Omega^1_A] \tag{7.11}$$

where Ω^1_A is the bimodule of noncommutative one-forms: it is generated by symbols $da, a \in A/k \cdot 1$, linear in a and subject to relations $d(ab) = da \cdot b + a \cdot db$. The differential B induces $B : A \to C^{\mathrm{sh}}_1(A)$ which under (7.11) becomes the De

Rham differential $d : a \mapsto da$. Factoring out all components $u^p C_n(A)$, $n \geq 2$, and $u^p bC_1(A)$, we get the short cyclic complexes

$$\mathrm{CC}^{\mathrm{sh}}_\bullet(A),\ \mathrm{CC}^{-,\mathrm{sh}}_\bullet(A),\ \mathrm{CC}^{\mathrm{per},\mathrm{sh}}_\bullet(A) \tag{7.12}$$

Dually, for a coalgebra B define

$$C^0_{\mathrm{II},\mathrm{sh}}(B) = B;\ C^1_{\mathrm{II},\mathrm{sh}}(B) = \ker(C^1_{\mathrm{II}}(B) \xrightarrow{b} C^2_{\mathrm{II}}(B));$$

$C^n_{\mathrm{II},\mathrm{sh}}(B) = 0$ for $n \geq 2$. Replacing $\mathrm{C}^\bullet_{\mathrm{II}}$ by $\mathrm{C}^\bullet_{\mathrm{II},\mathrm{sh}}$ in (7.5), (7.6), (7.7), we get complexes

$$\mathrm{CC}^\bullet_{\mathrm{II},\mathrm{sh}}(B),\ \mathrm{CC}^\bullet_{-,\mathrm{II},\mathrm{sh}}(B),\ \mathrm{CC}^\bullet_{\mathrm{per},\mathrm{II},\mathrm{sh}}(B). \tag{7.13}$$

Lemma 7.7. *If A is free as an algebra then the projections*

$$\mathrm{C}_\bullet(A) \to \mathrm{C}^{\mathrm{sh}}_\bullet(A)$$

and

$$\mathrm{CC}^?_\bullet(A) \to \mathrm{CC}^{?,\mathrm{sh}}_\bullet(A)$$

are quasi-isomorphisms. Dually, if B is cofree as a coalgebra then the embeddings

$$\mathrm{C}^\bullet_{\mathrm{II}}(B) \to \mathrm{C}^\bullet_{\mathrm{II},\mathrm{sh}}(B)$$

$$\mathrm{CC}^\bullet_{?,\mathrm{II}}(B) \to \mathrm{CC}^\bullet_{?,\mathrm{II},\mathrm{sh}}(B)$$

are quasi-isomorphisms. Here, as above, ? may stand for $-$, per, *or nothing.*

Now apply the lemma to $B = \mathrm{Bar}(A)$ and observe that:
$\mathrm{C}^\bullet_{\mathrm{II},\mathrm{sh}}$ is isomorphic to the cone of

$$1 - \tau : (A^{\otimes \bullet+1}, b') \to (A^{\otimes \bullet+1}, b);$$

$\mathrm{CC}^\bullet_{\mathrm{II},\mathrm{sh}}$ is isomorphic to the standard $(b, b', 1-\tau, N)$ complex for computing negative cyclic homology; $\mathrm{CC}^\bullet_{\mathrm{perII},\mathrm{sh}}$ is isomorphic to the standard $(b, b', 1-\tau, N)$ complex for computing periodic cyclic homology. This proves Theorem 7.6.

7.6. The big Hochschild and cyclic complexes.

Definition 7.8. Put

$$\mathrm{C}^{\mathrm{big}}_\bullet(A) = \mathrm{C}^{-\bullet}_{\mathrm{II}}(\mathrm{Bar}(A))$$

$$\mathrm{CC}^{\mathrm{big}}_\bullet(A) = \mathrm{CC}^{-\bullet}_{-,\mathrm{II}}(\mathrm{Bar}(A))$$

$$\mathrm{CC}^{\mathrm{NEG}}_\bullet(A) = \mathrm{CC}^{-\bullet}_{\mathrm{II}}(\mathrm{Bar}(A))$$

$$\mathrm{CC}^{\mathrm{PER}}_\bullet(A) = \mathrm{CC}^{-\bullet}_{\mathrm{per},\mathrm{II}}(\mathrm{Bar}(A))$$

Due to Theorem 7.6, these complexes are natural deformation retracts of their standard counterparts. When A is an ordinary algebra then

$$\mathrm{C}^{\mathrm{big}}_\bullet(A) = 0,\ \bullet < 0;$$

$$\mathrm{C}^{\mathrm{big}}_0(A) \xrightarrow{\sim} \bigoplus_{k=0}^{\infty} A^{\otimes(k+1)};$$

$$\mathrm{C}^{\mathrm{big}}_n(A) \xrightarrow{\sim} \bigoplus_{k=0}^{\infty} (A^{\otimes(k+1)})^{m_k}$$

for some integers m_k (dependent on n). Similarly to the standard case, one has

$$\mathrm{CC}^{\mathrm{NEG}}_\bullet(A) = \mathrm{C}^{\mathrm{big}}_\bullet(A)[[u]];\ \mathrm{CC}^{\mathrm{PER}}_\bullet(A) = \mathrm{C}^{\mathrm{big}}_\bullet(A)((u));$$

$$\mathrm{CC}^{\mathrm{big}}_{\bullet}(A)=\mathrm{C}^{\mathrm{big}}_{\bullet}(A)((u))/u\mathrm{C}^{\mathrm{big}}_{\bullet}(A)[[u]]$$

The differential is of the form $\mathbf{b}+u\mathbf{B}$.

THEOREM 7.9. *There is a natural action of the DG algebra $\mathcal{A}_1(\mathfrak{g}_A)$ on the big negative cyclic complex $\mathrm{CC}^{\mathrm{NEG}}_{\bullet}(A)$.*

PROOF. The action is defined by operators dual to the ones in Definition 6.3. □

7.6.1. *Extended noncommutative Cartan calculus for the short cochain complex.* In the rest of this section we will briefly outline another way to extend the Cartan calculus. Namely, we will observe that it extends to the short cochain complex $\mathfrak{g}^{\mathrm{sh}}_A$. After that, one can replace A by its cofibrant DG resolution R; the inclusion $\mathfrak{g}^{\mathrm{sh}}_R\to\mathfrak{g}_R$ is a quasi-isomorphism; it is also known that $\mathfrak{g}_R$ and $\mathfrak{g}_A$ are equivalent as DG Lie algebras. This yields an action of a DG Lie algebra equivalent to $\mathfrak{g}_A$ on another version of a big negative cyclic complex, namely $\mathrm{CC}^{-}_{\bullet}(R)$. We are not using this anywhere else in the article.

DEFINITION 7.10. Let A be a unital algebra. Define a differential graded Lie subalgebra of $\mathfrak{g}_A$ (consisting of zero-cochains and one-cocycles)

$$\mathfrak{g}^{\mathrm{sh}}_A=A[1]\rtimes Der(A)$$

Here $A[1]$ is Abelian. The differential is zero on $\mathrm{Der}(A)$ and sends $a\in A[1]$ to $\mathrm{ad}(a)$.

PROPOSITION 7.11. *The action from Proposition 6.2 naturally extends to an action of the algebra $U(\mathfrak{g}^{\mathrm{sh}}_A)\ltimes_1\mathrm{Cobar}(U^{+}(\mathfrak{g}^{\mathrm{sh}}_A))$ on $\mathrm{CC}^{-}_{\bullet}(A)$.*

PROOF. For $x\in A$, we define

$$\lambda_x(a_1\otimes\ldots\otimes a_n)=\sum_{j=0}^{n}\pm a_1\otimes\ldots\otimes a_j\otimes x\otimes a_{j+1}\otimes\ldots\otimes a_n; \tag{7.14}$$

$$L_x(a_0\otimes\ldots\otimes a_n)=\pm a_0\otimes\lambda_x(a_1\otimes\ldots\otimes a_n); \tag{7.15}$$

$$\iota_x(a_0\otimes\ldots\otimes a_n)=a_0x\otimes a_1\otimes\ldots\otimes a_n \tag{7.16}$$

For $X_1,\ldots,X_m\in\mathfrak{g}^{\mathrm{sh}}_A$ and $X=X_1\ldots X_m\in U(\mathfrak{g}^{\mathrm{sh}}_A)$, put

$$L_X=L_{X_1}\ldots L_{X_m} \tag{7.17}$$

Extend the composition operation from $\mathrm{Der}(A)$ to $\mathfrak{g}^{\mathrm{sh}}_A$ as follows. For $D\in\mathrm{Der}(A)$ and $x\in A$, $D\circ x=D(x)$ and $x\circ D=0$; for $x,y\in A$, $x\circ y=0$. For $X_1,\ldots,X_m\in\mathfrak{g}^{\mathrm{sh}}_A$ and $X=X_1\ldots X_m\in U^{+}(\mathfrak{g}^{\mathrm{sh}}_A)$, put

$$\overline{X}=\overline{X_1\ldots X_m}=((\ldots(X_1\circ X_2)\circ\ldots)\circ X_m) \tag{7.18}$$

For $X_1,\ldots,X_m\in\mathfrak{g}^{\mathrm{sh}}_A$ and $X=X_1\ldots X_m\in U^{+}(\mathfrak{g}^{\mathrm{sh}}_A)$ set

$$I_X=\iota_{\bar{X}}+uS(X) \tag{7.19}$$

where

$$S_X(a_0\otimes\ldots\otimes a_n)=\sum_{j=0}^{n}(-1)^j 1\otimes a_j\otimes\ldots\otimes a_n\otimes a_0\otimes\lambda_X(a_1\otimes\ldots\otimes a_{j-1}) \tag{7.20}$$

Note that $\overline{X}$ is either a linear mar $A\to A$ (if all X_i are derivations) or an element of A. In the former case $\iota_{\overline{X}}$ is defined by (6.10), in the latter case by (7.16).

Let $X \in U(\mathfrak{g}_A^{\rm sh})$ act by L_X (7.17) and let (X), $X \in U^+(\mathfrak{g}_A^{\rm sh})$, act by I_X (7.19). We claim that $U(\mathfrak{g}_A^{\rm sh}) \ltimes_1 \mathrm{Cobar}(U^+(\mathfrak{g}_A^{\rm sh}))$ defines an action of $U(\mathfrak{g}_A^{\rm sh}) \ltimes_1 \mathrm{Cobar}(U^+(\mathfrak{g}_A^{\rm sh}))$. The proof will be sketched below. $\square$

8. The Gauss-Manin superconnection

8.1. The universal formulas. Recall the algebras

$$\mathcal{A}_0(\mathfrak{g}) = U(\mathfrak{g}) \ltimes_0 \mathrm{Cobar}(\mathrm{Sym}(\mathfrak{g})^+) \tag{8.1}$$

$$\mathcal{A}_1(\mathfrak{g}) = U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(U(\mathfrak{g})^+) \tag{8.2}$$

defined in 6.1. Those are algebras over $k[u]$; as we have seen, $\mathcal{A}_1(\mathfrak{g}_A)$ acts on the big negative cyclic complex of A. This action obviously extends to the (u)-adic completion of $\mathcal{A}_1(\mathfrak{g}_A)$. However, for applications we will rather need the (u^{-1})-adic completion of the localized algebra $\mathcal{A}_0[u^{-1}]$.

LEMMA 8.1. *Let k contain the rationals. There is a natural isomorphism of DG algebras*

$$\mathcal{A}_0(\mathfrak{g}) \xrightarrow{\sim} \mathcal{A}_1(\mathfrak{g})$$

PROOF. For $D_1, \ldots, D_k$ in $\mathfrak{g}$ define $(D_1 \ldots D_k)$ to be the coefficient at $t_1 \ldots t_k$ in

$$\frac{1}{k!} J(t_1 D_1 + \ldots + t_k D_k) \tag{8.3}$$

where J is as in Lemma 4.5. Here $t_1, \ldots, t_k$ are formal variables that are central. We view $(D_1 \ldots D_k)$ as an element of $\mathcal{A}_0\mathfrak{g}$. We get an $\mathfrak{g}$-equivariant map $\mathrm{Sym}(\mathfrak{g})^+ \to \mathcal{A}_1(\mathfrak{g})$ that extends to a DG algebra isomorphism $\mathcal{A}_0(\mathfrak{g}) \xrightarrow{\sim} \mathcal{A}_1(\mathfrak{g})$. $\square$

Let $\mathfrak{a}_n$ be the free graded Lie algebra with generators $\lambda_0, \ldots, \lambda_n$ of degree one.

LEMMA 8.2. *Let k be of characteristic zero. Then $k[u, u^{-1}] \to \mathcal{A}_0(\mathfrak{a}_n)[u^{-1}]$ is a quasi-isomorphism.*

PROOF. The cohomology of the differential $\partial_{\mathrm{Cobar}}$ is

$$\mathrm{Sym}(\mathfrak{a}_n[-1]) \otimes U(\mathfrak{a}_n)[u, u^{-1}] \tag{8.4}$$

The differential induced by B_0 is the standard Chevalley-Eilenberg differential computing the homology $H_\bullet(\mathfrak{a}_n, U(\mathfrak{a}_n))$ which is isomorphic to k. The corresponding spectral sequence proves the statement. $\square$

DEFINITION 8.3. Let $\widehat{\mathcal{A}}_0(\mathfrak{a}_n)$ be the completion of $\mathcal{A}_0(\mathfrak{a}_n))[u^{-1}]$ with respect to the increasing filtration induced by the grading of $\mathfrak{a}_n$. Define

$$D_{\lambda_j} = \lambda_j - J(\frac{1}{u} R_j) = \lambda_j - (\exp(\frac{R_j}{u}) - 1) \in \widehat{\mathcal{A}}_0(\mathfrak{a}_n)^1$$

where $R_j = \lambda_j^2$.

LEMMA 8.4. *Let k be of characteristic zero.*

(1) *For any j one has*

$$D_{\lambda_j}^2 = 0$$

(2) *There exist elements* $T(\lambda_0,\dots,\lambda_m)$ *of degree* $1-m$ *in* $\widehat{\mathcal{A}}_0(\mathfrak{a}_m)$ *for all* $m>1$ *such that*

$$(\partial_{\mathrm{Cobar}}+uB_0)T(\lambda_0,\dots,\lambda_n)+D_{\lambda_0}T(\lambda_0,\dots,\lambda_n)+$$
$$\sum_{j=1}^{n-1}(-1)^{j-1}T(\lambda_0,\dots,\lambda_j)T(\lambda_j,\dots,\lambda_n)+(-1)^nT(\lambda_0,\dots,\lambda_n)D_{\lambda_n}$$
$$+\sum_{j=1}^{n-1}T(\lambda_0,\dots,\widehat{\lambda}_j,\dots,\lambda_n)=0$$

Proof. Follows from Lemma 8.2. Indeed, let us look for

$$T(\lambda_0,\lambda_1)=1+\sum_{k=1}^{\infty}T_k(\lambda_0,\lambda_1) \tag{8.5}$$

where T_k is homogenous of degree k. By induction in k, we find T_k as the solution for $(\partial_{\mathrm{Cobar}}+uB_0)T_k=U_k$ where U_k is some given cocycle. This gives us $T(\lambda_0,\lambda_1)$. Similarly, assume we have found all $T(\lambda_0,\dots,\lambda_m)$ for $m<n$.

$$T(\lambda_0,\dots,\lambda_n)=\sum_{k=1}^{\infty}T_k(\lambda_0,\dots,\lambda_n) \tag{8.6}$$

Then we can find each homogenous component T_k using induction in k. □

8.2. The main statement. Let S be an algebraic variety over a unital commutative ring k. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$-algebras. We assume that locally in S there is a connection

$$\nabla:\mathcal{A}\to\Omega^1_S\otimes_{\mathcal{O}_S}\mathcal{A} \tag{8.7}$$

on the family $\mathcal{A}$. We do not require the connection to preserve the algebra structure.

Let $\mathrm{CC}^{\mathrm{PER}}_\bullet(\mathcal{A}/\mathcal{O}_S)$ be the sheaf of big periodic cyclic complexes where $\mathcal{O}_S$ is regarded as the ring of scalars. Then ∇ extends to an operator of degree one on $\Omega^\bullet_S\otimes_{\mathcal{O}_S}\mathrm{CC}^{\mathrm{PER}}_\bullet(\mathcal{A}/\mathcal{O}_S)$.

Theorem 8.5. *Let k be of characteristic zero. Let S be an algebraic variety over k and let $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$-algebras.*

(1) *For any connection ∇ on $\mathcal{A}$ there exists canonically defined element of degree 1*
$$A(\nabla)\in\Omega^\bullet_S\otimes_{\mathcal{O}_S}\mathrm{End}^\bullet_{\mathcal{O}_S}(\mathrm{CC}^{\mathrm{PER}}_\bullet(\mathcal{A}/\mathcal{O}_S))$$
such that
$$(\nabla^{\mathrm{GM}})^2=0$$
where $\nabla^{\mathrm{GM}}=\mathbf{b}+u\mathbf{B}+\nabla+A(\nabla)$.

(2) *For any connections $\nabla_0,\dots,\nabla_n$ on $\mathcal{A}$, there is a canonically defined element of degree $2-n$*
$$T(\nabla_0,\dots,\nabla_n)\in\Omega^\bullet_S\otimes_{\mathcal{O}_S}\mathrm{End}^\bullet_{\mathcal{O}_S}(\mathrm{CC}^{\mathrm{PER}}_\bullet(\mathcal{A}/\mathcal{O}_S))$$
such that
$$(\mathbf{b}+u\mathbf{B})T(\nabla_0,\dots,\nabla_n)+\nabla_0^{\mathrm{GM}}T(\nabla_0,\dots,\nabla_n)+$$
$$\sum_{j=1}^{n-1}(-1)^jT(\nabla_0,\dots,\nabla_j)T(\nabla_j,\dots,\nabla_n)+(-1)^nT(\nabla_0,\dots,\nabla_n)\nabla_n^{\mathrm{GM}}=0$$

Proof. Put

$$R_\nabla = \nabla m + \nabla^2; \tag{8.8}$$

then

$$(\mathbf{b} + u\mathbf{B} + \nabla)^2 = L_{R_\nabla} \tag{8.9}$$

where m is the Hochschild cochain defining the product on $\mathcal{A}$.

Now put

$$\nabla^{\mathrm{GM}} = \mathbf{b} + u\mathbf{B} + \nabla - J(\frac{1}{u}R_\nabla) \tag{8.10}$$

and

$$T(\nabla_0, \ldots, \nabla_n) = T(m + \nabla_0, \ldots, m + \nabla_n) \tag{8.11}$$

as in Lemma 8.4. The same lemma implies the statement. □

8.3. Explicit formulas and integrality. Lemma 8.4 can be improved to provide explicit formulas for $T(\lambda_0, \ldots, \lambda_n)$. Those formulas have an integrality property; namely, if $\lambda_0, \ldots, \lambda_n$ are divisible by a prime $p > 3$ then $T(\lambda_0, \ldots, \lambda_n)$ are defined in the p-adic completion of $\mathcal{A}_0(\mathfrak{a}_n)$ (as are D_{λ_i}). This is not surprising. Indeed, in the proof of the lemma we used the fact that in characteristic zero, both the cobar construction and the chain complex $C_\bullet(\mathfrak{a}_n, U(\mathfrak{a}_n))$ are (almost) acyclic. In fact this is true over the integers for certain divided power versions of these complexes. We were not able to deduce the p-adic analog of Lemma 8.4 just from that observation. Instead, it is possible to follow this logic more closely and to write explicit formulas. These formulas are themselves of interest. They resemble certain WKB-type expansions in a Fock space, figuring fast oscillating exponentials where the Planck constant $\hbar$ is replaced by the formal parameter u of degree two. We will give details in the forthcoming article.

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